A Characterisation of Optimal Channel Assignments for Wireless Networks Modelled as Cellular and Square Grids

Shashanka MVS, Amrita Pati
Birla Institute of Technology & Science
Pilani, India
fd99154@lycos.com, amrita013@yahoo.com

Anil M. Shende
Roanoke College
Salem, VA, USA
shende@roanoke.edu

Abstract

In this paper we first present a uniformity property that characterises optimal channel assignments for networks arranged as cellular or square grids. Then, we present optimal channel assignments for cellular and square grids; these assignments exhibit a high value for δ_1 — the separation between channels assigned to adjacent stations. Based on empirical evidence, we conjecture that the value our assignments exhibit is an upper bound on δ_1.

1 Introduction

The enormous growth of wireless networks has made the efficient use of the scarce radio spectrum important. A "Frequency Assignment Problem" (FAP) models the task of assigning frequencies (channels) from a radio spectrum to a set of transmitters and receivers, satisfying certain constraints [6]. The main difficulty in an efficient use of the radio spectrum is the *interference* caused by unconstrained simultaneous transmissions. Interferences can be eliminated (or at least reduced) by means of suitable *channel assignment* techniques, which partition the given radio spectrum into a set of disjoint channels that can be used simultaneously by the stations while maintaining acceptable radio signals. Since radio signals get attenuated over distance, two stations in a network can use the same channel without interferences provided the stations are spaced sufficiently apart. Stations that use the same channel are called *co-channel stations*. The minimum distance at which channels can be reused with no interferences is called the *co-channel reuse distance* (or simply *reuse distance*) and is denoted by σ.

In a *dense* network – a network where there are a large number of transmitters and receivers in a small area – interference is more likely. Thus, reuse distance needs to be high in such networks. Moreover, channels assigned to nearby stations must be separated in value by at least a gap which is inversely proportional to the distance between the two stations. A minimum *channel separation* δ_i is required between channels assigned to stations at distance i, with $i < \sigma$, such that δ_i decreases when i increases [5]. The purpose of channel assignment algorithms is to assign channels to transmitters in such a way that (1) the co-channel reuse distance and the channel separation constraints are satisfied, and (2) the *span* of the assignment, defined to be the difference between the highest and the lowest channels assigned, is as small as possible [1].

This paper has two significant contributions:

1. A characterisation of optimal colourings for cellular and square grids. We essentially show a nice uniformity across the grid that every optimal colouring must satisfy. (See Section 2.)

2. We present optimal channel assignments for cellular and square grids where the channel separation between adjacent stations is large. Empirical evidence suggests that the separation that our assignments realise is an upper bound on the separation that *any* optimal channel assignment scheme for these grids can achieve. (See Section 3.)

2 A Characterisation of Optimal Colourings

We first introduce square grids and cellular grids. We explain tilings in both grids, and define some notation. Then we present our characterisation of optimal colourings in cellular and square grids.

Given a graph, G, for a network, and a reuse distance σ, consider the augmented graph obtained from G by adding edges between all those pairs of vertices that are at a distance of at most $\sigma - 1$. Clearly, then, the size (number of vertices) of any clique in this augmented graph places a lower bound on an $L(1, \vec{1}_{\sigma-1})$ colouring (and hence, on an $L(\delta_1, \ldots, \delta_{\sigma-1})$ colouring) for G; the best such lower

0-7695-1926-1/03/$17.00 (C) 2003 IEEE
Proceedings of the International Parallel and Distributed Processing Symposium (IPDPS'03)

bound is given by the size of a maximum clique in the augmented graph.

In sections 2.1 and 2.2 below we treat cellular and square grids, respectively, as graphs. Given σ, a *tile* is a clique in the augmented graph as described above. In each case, we will denote the number of vertices in a tile by $c(\sigma)$ (to avoid notational clutter, we do not include the graph as a parameter to c).

2.1 Cellular Grids

Lemma 1. *The number of vertices in a tile corresponding to reuse distance σ, denoted by $c(\sigma)$ is given by $c(\sigma) = (3\sigma^2 + (\sigma \bmod 2))/4$.* □

For a particular σ, tiles are hexagons and they tile the entire grid $\mathbf{A}_2$. The hexagons are regular with sides of $\lceil \frac{\sigma}{2} \rceil$ vertices for σ odd. In case of σ even, alternate sides of the hexagon are equal, consecutive sides having $\frac{\sigma}{2}$ and $\lceil \frac{\sigma+1}{2} \rceil$ vertices respectively.

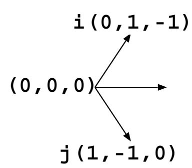

Figure 1. Basis vectors in $\mathbf{A}_2$

Figure 1 shows the coordinate system we use for representing vertices in $\mathbf{A}_2$ where $(0, 1, -1)$ and $(1, -1, 0)$ indicate the basis vectors i and j. We place a hexagon with one of the vertices at $(0, 0)$ and call it a *canonical tile*.

In the table below, we list the coordinates of the corners of a hexagon in clockwise order, for various values of σ.

σ mod $4 = 0$	σ mod $4 = 2$	σ mod $4 = 1, 3$
$(0, 0)$	$(0, 0)$	$(0, 0)$
$(\frac{\sigma}{2} - 1, 0)$	$(\frac{\sigma}{2}, 0)$	$(\lfloor \frac{\sigma}{2} \rfloor, 0)$
$(\sigma - 1, \frac{\sigma}{2})$	$(\sigma - 1, \frac{\sigma}{2} - 1)$	$(\sigma - 1, \lfloor \frac{\sigma}{2} \rfloor)$
$(\sigma - 1, \sigma - 1)$	$(\sigma - 1, \sigma - 1)$	$(\sigma - 1, \sigma - 1)$
$(\frac{\sigma}{2} - 1, \sigma - 1)$	$(\frac{\sigma}{2}, \sigma - 1)$	$(\lfloor \frac{\sigma}{2} \rfloor, \sigma - 1)$
$(0, \frac{\sigma}{2})$	$(0, \frac{\sigma}{2} - 1)$	$(0, \lfloor \frac{\sigma}{2} \rfloor)$

Hexagons with origin other than $(0, 0)$ will be referred to as *tiles*. The six hexagons with their origins at $(-\sigma, -\lceil \frac{\sigma}{2} \rceil)$, $(-\lceil \frac{\sigma}{2} \rceil, \lfloor \frac{\sigma}{2} \rfloor)$, $(\lfloor \frac{\sigma}{2} \rfloor, \sigma)$, $(\sigma, \lceil \frac{\sigma}{2} \rceil)$, $(\lceil \frac{\sigma}{2} \rceil, -\lfloor \frac{\sigma}{2} \rfloor)$ and $(-\lfloor \frac{\sigma}{2} \rfloor, -\sigma)$ tile $\mathbf{A}_2$ around the canonical one. The tiling is shown for $\sigma = 4$ and $\sigma = 5$ in Figure 2. The points marked in the figure correspond to the origins of the tiles. The canonical tile is marked as H_0 and the tiles surrounding it are marked H_1 to H_6. We name the edges of the tiles as follows: the edge of tile H_0 which is adjacent to the tile H_i will be denoted by t_i.

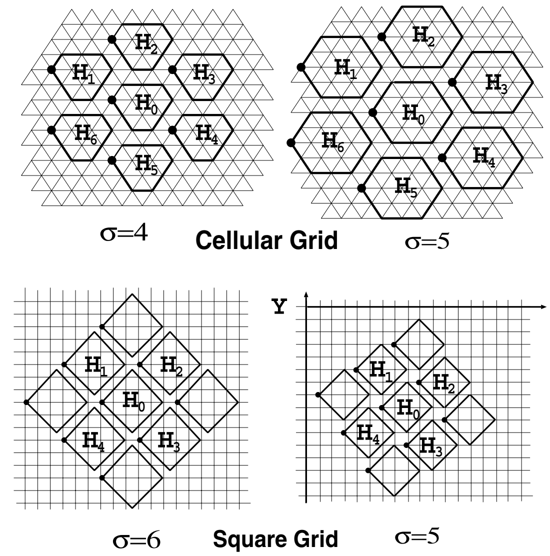

Figure 2. Tiling of $\mathbf{A}_2$ and $\mathbf{Z}^2$

2.2 Square Grids

Lemma 2. *The number of vertices in a tile corresponding to reuse distance σ, denoted by $c(\sigma)$ is given by $c(\sigma) = \lceil \frac{\sigma^2}{2} \rceil$.* □

For a particular σ, tiles are squares with their diagonals along the X and Y axes and every side containing $\lceil \frac{\sigma}{2} \rceil$ vertices. They tile the entire grid $\mathbf{Z}^2$. In the case of odd σ, every corner of a tile corresponds to a vertex whereas in the case of even σ, opposite corners along the X direction correspond to vertices.

We place a square with one of its vertices at $(0, 0)$ and call it a *canonical tile*. Squares with origins other than $(0, 0)$ will just be referred to as *tiles*. The four squares with origins at $(-\lfloor \frac{\sigma}{2} \rfloor, \lceil \frac{\sigma}{2} \rceil)$, $(\lceil \frac{\sigma}{2} \rceil, \lfloor \frac{\sigma}{2} \rfloor)$, $(\lfloor \frac{\sigma}{2} \rfloor, -\lceil \frac{\sigma}{2} \rceil)$ and $(-\lceil \frac{\sigma}{2} \rceil, -\lfloor \frac{\sigma}{2} \rfloor)$ tile $\mathbf{Z}^2$ around the canonical one. The tiling is shown for $\sigma = 5$ and $\sigma = 6$ in Figure 2. The points marked in the figure correspond to the origins of tiles. The canonical tile is marked as H_0 and the neighboring tiles are marked H_1 to H_4. We name the edges of the tiles as follows: the edge of tile H_0 which is adjacent to the tile H_i will be denoted by t_i.

There is another kind of tiling possible in both cellular and square grids for odd reuse distances, as shown in Figure 3 for $\sigma = 5$. We shall refer to the tiling introduced earlier as *Tiling A* and the one shown in Figure 3 as *Tiling B*.

Definition 1. *In a cellular grid tile, we define a* diagonal *to be a line formed by all vertices having the same i coordinate. It is represented as L_i, where i, called the* diagonal number, *is the corresponding i coordinate.*

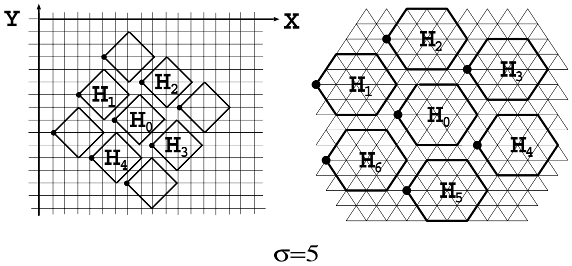

Figure 3. Possible tiling of $\mathbf{Z}^2$ and $\mathbf{A}_2$ for odd σ

Definition 2. *In a square grid tile, we define a* vertical *to be a line formed by all vertices having the same i coordinate. It is represented as V_i, where i, called the* vertical number, *is the corresponding i coordinate.*

Definition 3. *In a square grid tile, we define a* diagonal *to be a line of the form $i - j = c$ where c is a constant. It is represented as D_i where i, called the* diagonal number *is given by $(i - j)$ mod σ.*

In a tile corresponding to reuse distance σ, there are σ *diagonals/verticals* as the case may be. Figure 9 shows the *diagonals* in a cellular grid. Figures 10 and 11 show *verticals* and *diagonals* of a square grid respectively.

Definition 4.

1. *Consider a point p in a square/cellular grid and consider all points which are at a distance σ from p, where σ is the reuse distance. In the case of square grids, all these points form a square centered at p and in the case of cellular grids, they form a hexagon centered at p. This square/hexagon will be called the* bounding box surrounding point *p and will be denoted by $B(p)$. The edges, considered in a clockwise direction, are denoted by $d_1, d_2, \ldots, d_4$ in case of square grids and $d_1, d_2, \ldots, d_6$ in case of cellular grids as shown in Figure 4.*

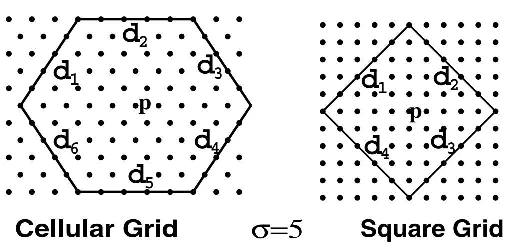

Figure 4. *Bounding Box $B(p)$* **with edges marked**

2. *Consider the* bounding box *for point p. Every edge contains $\sigma - 1$ vertices apart from the two corners. Each corner, which belongs to two edges $d_{\overline{i}}$ and $d_{\overline{i+1}}$, is taken to be a part of the second edge $d_{\overline{i+1}}$, where $\overline{i}$ refers to (i mod s), s being 4 in case of square grids and 6 in case of cellular grids. We number each vertex of all edges consecutively starting with 1 being assigned to the left-corner vertex. These numbers are called as* position numbers. *This is shown in Figures 5, 6, 7 and 8.*

2.3 Optimal Colouring Schemes

A colouring scheme is optimal if it uses the smallest possible number of colours. In other words, a colouring which uses colours from the set $\{0, 1, \ldots, g\}$ will be optimal if it uses the smallest possible value for g. From Lemma 1 and Lemma 2, we know that $c(\sigma)$ is a lower bound on the number of colours used. We are concerned only with such colouring schemes which use only $c(\sigma)$ different colours.

We already know that σ is the *minimum* distance at which channels can be reused. In other words, the same colour can be used for vertices which are at distance σ or greater. The following lemma establishes that in an optimal colouring the nearest vertex where a colour is reused is no more than distance σ away.

Lemma 3. *Consider an optimal colouring scheme for a wireless network model with reuse distance σ. For a given point p, there exists at least one point at distance σ from p which has the same colour as p.*

Proof. Let us assume that, on the contrary, there is no point at distance σ from p which has the same colour as p. Thus, no point inside, or on the boundary of, $B(p)$ is assigned the same colour as that of p.

Now, consider one of the edges of $B(p)$, say d_1 and a tile inside $B(p)$ such that one of its edges (t_1) is a completely contained in this edge of $B(p)$. Clearly, p is not in this tile. Since we have an optimal colouring, one of the points in the tile must be assigned the same colour as the colour assigned to p. This is a contradiction, and hence the result. □

We now present a theorem using which we will be able to establish an important property of optimal colouring schemes.

Theorem 1. *Consider an optimal colouring scheme for a wireless network model with reuse distance σ. For every point p, there is a position number n, such that each point corresponding to this position number on each edge of the bounding box surrounding p has the same colour as p. Moreover, $n = \lceil \frac{\sigma}{2} \rceil$ or $n = \lceil \frac{\sigma}{2} \rceil + 1$*

Proof. Consider the edge d_1 of the bounding box around p, $B(p)$. Consider the k different tiles, each of whose edges t_1

is a part of the edge d_1 of $B(p)$, where $\sigma = 2k+1$ for odd σ and $\sigma = 2k$ for even σ. Refer Figures 5, 6, 7, 8.

Let $P(i)$ denote the sequence of position numbers on d_1 of $B(p)$ that are on the edges t_1 of the i-th of these k tiles. In the case of odd σ, $P(i)$ are given by:

$$\begin{aligned} P(1) &= \langle 3, \ldots, k+2 \rangle \\ P(2) &= \langle 3, 4, \ldots, k+3 \rangle \\ &\vdots \quad \vdots \\ P(k) &= 2+(k-1), 2+k, \ldots, 2k+1 \end{aligned} \qquad (1)$$

In case of even σ, $P(i)$ are given by:

$$\begin{aligned} P(1) &= 2, 3, \ldots, k+1 \\ P(2) &= 3, 4, \ldots, k+2 \\ &\vdots \quad \vdots \\ P(k) &= 2+(k-1), 2+k, \ldots, 2k \end{aligned} \qquad (2)$$

Since the colouring is optimal, the colour c, that the point p is coloured in, must appear somewhere on each of these tiles. Except for the edge t_1, each of these tiles is completely contained within $B(p)$. Thus, the colour c must appear on the edge t_1 of each of these tiles, otherwise, the reuse constraint is violated. Since no pair of vertices with position numbers 2, 3, $\ldots$, $2k+1$ on the edge d_1 of $B(p)$ are at a distance σ, it must be the case that the colour c is assigned to some vertex that is common to all the above tiles. In case of odd σ, as seen from Equation(1), the only two common vertices are the ones with position numbers $k+1$ and $k+2$, and hence, one of these two vertices must be assigned the colour c. In case of even σ, we see from Equation(2) that the only common vertex is the one corresponding to position number $k+1$, and hence, it has to be assigned the colour c. A similar argument establishes that on each edge of $B(p)$, the vertices corresponding to position numbers $k+1$ and $k+2$ in case of odd σ and $k+1$ in case of even σ are the only possible candidates for being assigned colour c.

Now, in case of odd σ, suppose the vertex corresponding to position number $k+1$ on the edge d_1 of $B(p)$ is assigned colour c. Let us name this vertex q. Suppose, by way of contradiction, the vertex with position number $k+2$ on the edge d_2 of $B(p)$ is assigned colour c. Let us name this vertex x. We will now consider cellular and square grids separately in two different cases.

Case 1 (Cellular Grids): Consider the bounding box $B(q)$. The edge d_3 of $B(q)$ passes through the vertex with position number $k+1$ on the edge d_2 of $B(p)$ as shown in Figure 6. By the above argument, one of the two vertices with position numbers $k+1$ or $k+2$ on this edge d_3 of $B(q)$ must be assigned colour c. But both these vertices are at a distance less than σ from the vertex x. Therefore, x cannot be assigned colour c implying that the vertex with position number $k+1$ on the edge d_2 must be assigned colour c.

Case 2 (Square Grids): Consider the bounding boxes $B(q)$ and $B(x)$. Let us name the point where the edges d_2 of $B(q)$ and d_1 of $B(x)$ intersect, r. If both q and x are coloured c, it follows that r should be assigned the colour c. This is not possible because r lies within the bounding box $B(p)$ of point p which is also coloured c. This implies that the vertex with position number $k+1$ on the edge d_2 must be assigned colour c.

Similar arguments in both cases above establish that, if the vertex with position number $k+1$ on any one edge of $B(p)$ is coloured the same as the colour of p, then on each edge of $B(p)$, the vertex with position number $k+1$ is coloured the same as p. □

Corollary 1. *For odd σ, if we choose vertices corresponding to position number $k+1$ on each edge of the bounding box of a particular point to be coloured the same as the point, the resulting colouring will match* Tiling B *(Figure 3). If the position number chosen is $k+2$, the resulting tiling is* Tiling A *(Figure 2).*

The following characterisation of optimal colourings of cellular and square grids is an immediate consequence of Therorem 1.

Theorem 2. *Given σ, and given a tiling of a cellular or square grid by tiles (for σ), a colouring with reuse distance σ is optimal iff all the tiles in the tiling are identical in their colour assignment.* □

3 Optimal $L(\delta_1, \vec{1}_{\sigma-2})$ Colouring

In this section, we deal with optimal frequency assignment schemes for wireless networks modelled as cellular grids and square grids. We first present an $L(\delta_1, \vec{1}_{\sigma-2})$

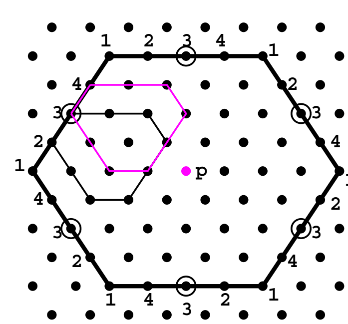

Figure 5. Cellular Grid *Bounding Box* **for** $\sigma = 4$ **with position numbers**

colouring scheme of $\mathcal{G}(\mathbf{A}_2)$ for the case where *reuse distance* is odd i.e., $\sigma = 2k+1$, $k \in \{1, 2, \ldots\}$. This is followed by an $L(\delta_1, \vec{1}_{\sigma-2})$ colouring scheme of $\mathcal{G}(\mathbf{Z}^2)$ for all values of σ.

3.1 Cellular Grids

We present a colouring scheme where δ_1 varies as the square of σ, for $\sigma \geq 5$, σ odd. We note that the colouring of the entire cellular grid is achieved by colouring one tile and reproducing the same colouring in all the tiles present in the grid. Recalling that the number of vertices $c(\sigma)$, in a tile corresponding to an odd reuse distance $\sigma = 2k+1$ is equal to $3k^2 + 3k + 1$, we observe the following properties of the tiling :

Lemma 4. *1. Colouring $c(\sigma)$ points starting from the vertex of a tile along the direction j is equivalent to colouring all the diagonals of a tile in the following order: $L_0, L_{k+1}, L_1, L_{k+2}, \ldots, L_{k-1}, L_{2k}, L_k$.*

2. Along a line $i = m$, where m is a constant, any pair of points which are at a distance $c(\sigma)$ apart will have the same colour assigned to them.

3. Consider a point (p, q) on the line $i = p$. The point $(p+1, q-3k-1)$ on the line $i = p+1$ will have the same colour as (p, q). □

From Lemma 4.1, we see that a colouring for $c(\sigma)$ points along a line $i = m$ for some arbitrary m describes the colouring for the entire grid.

To colour along the line $i = 0$, we proceed as follows: Starting with the point $(0, 0)$ which is assigned the colour 0, we assign consecutive colours to every third vertex, and wrap around after the $c(\sigma)$th vertex. This will colour all the $c(\sigma)$ points in three passes uniquely. This can be easily seen because $c(\sigma) = c'(k) = 3k^2 + 3k + 1$ is 1 mod 3. Consecutive sets of $c(\sigma)$ vertices along this line follow the same colouring pattern.

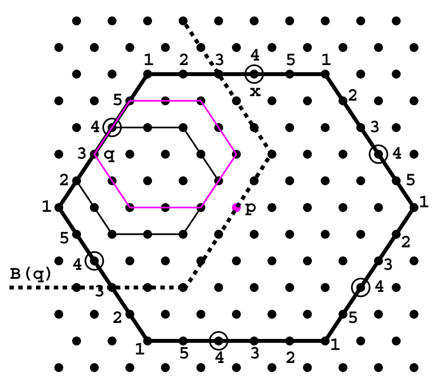

Figure 6. Cellular Grid *Bounding Box* **for** $\sigma = 5$ **with position numbers**

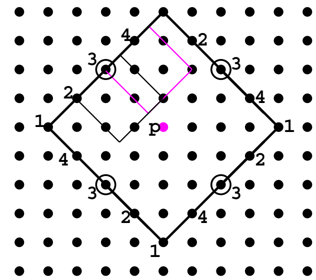

Figure 7. Square Grid *Bounding Box* **for** $\sigma = 4$ **with position numbers**

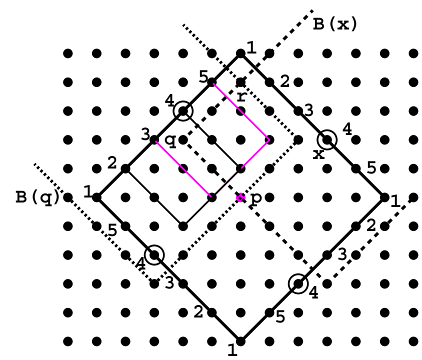

Figure 8. Square Grid *Bounding Box* **for** $\sigma = 5$ **with position numbers**

Mathematically, this colouring scheme can be expressed as follows. Let $\chi(i, j)$ represent the colour assigned to the vertex (i, j) and $\chi'(j)$ represent the colour assigned to the vertex $(0, j)$, i.e. $\chi'(j) = \chi(0, j)$. We first give a formula for $\chi'(j)$ and then derive an expression for $\chi(i, j)$.

$$\chi'(j) = \begin{cases} \rho, & \bar{j} = 0 \bmod 3 \\ \rho + 2k^2 + 2k + 1, & \bar{j} = 1 \bmod 3 \\ \rho + k^2 + k + 1, & \bar{j} = 2 \bmod 3 \end{cases}$$

where $\bar{j} = j \bmod c(\sigma)$ and $\rho = \lfloor \frac{\bar{j}}{3} \rfloor$. Now, from Lemma 4.3, we can easily derive that

$$\chi(i, j) = \chi'\big(j + i(3k+1)\big)$$

From the above formula, it can be easily seen that given any arbitrary point (i, j) in the grid, the colour assigned to (i, j) can be computed in constant time.

Theorem 3. *For all $\sigma = 2k+1, k = \{1, 2, \ldots\}$, the colouring scheme described above is an optimal $L(\delta_1, \vec{1}_{\sigma-2})$-colouring for $\mathcal{G}(\mathbf{A}_2)$, with $\delta_1 = k^2$.*

Proof. From Lemma 1, $c(\sigma)$ is a lower bound. We can easily see that each vertex in the tile is assigned a unique colour from the set $\{0, 1, \ldots, c(\sigma) - 1\}$. This implies that the optimality condition is satisfied.

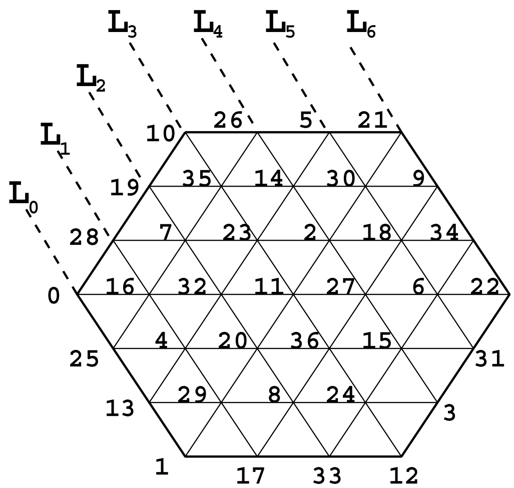

Figure 9. $L(\delta_1, \vec{1}_{\sigma-2})$ **colouring for** $\sigma = 7$

Again, the above scheme ensures that corresponding points in neighboring tiles have the same colour and are exactly σ distance apart. Thus, the re-use constraint is satisfied.

To derive the value of δ_1, we proceed as follows. Consider a point (i, j) in the grid. Its six neighbors are $(i, j-1)$, $(i+1, j)$, $(i+1, j+1)$, $(i, j+1)$, $(i-1, j)$ and $(i-1, j-1)$.

The colour assigned to (i, j) according to the above scheme will be $\chi(i, j) = \chi'\big(j + i(3k+1)\big)$.

The following table shows the colours assigned to the neighbors of (i, j). (All the colour expressions are modulo $c(\sigma)$.)

$(i, j-1)$	$\big(\chi(i, j) + k^2 + k\big)$
$(i+1, j)$	$\big(\chi(i, j) + 2k^2 + 3k + 1\big)$
$(i+1, j+1)$	$\big(\chi(i, j) + k^2 + 2k + 1\big)$
$(i, j+1)$	$\big(\chi(i, j) + 2k^2 + 2k + 1\big)$
$(i-1, j)$	$\big(\chi(i, j) + k^2\big)$
$(i-1, j-1)$	$\big(\chi(i, j) + 2k^2 + k\big)$

From the above table, we see that the least difference between the colours assigned to neighbouring points is k^2. Hence, $\delta_1 = k^2$. □

3.2 Square Grids

We present colouring schemes where δ_1 varies as the square of σ, for $\sigma \geq 4$. There are two different schemes, one for the case where σ is odd and one for even σ. We note that the colouring of the entire square grid is achieved by colouring one tile and reproducing the same colouring in all the tiles present in the grid.

Odd σ

Recalling that the number of vertices $c(\sigma)$, in a tile corresponding to an odd reuse distance $\sigma = 2k + 1$ is equal to $2k^2 + 2k + 1$, we observe the following properties of the tiling :

Lemma 5. *1. Colouring $c(\sigma)$ points starting from the vertex of a tile along the direction j is equivalent to colouring all the diagonals of a tile in the following order: $V_0, V_k, V_{2k}, V_{k-1}, \ldots, V_{k+2}, V_1, V_{k+1}$.*

2. Along a line $i = m$, where m is a constant, any pair of points which are at a distance $c(\sigma)$ apart will have the same colour assigned to them.

3. Consider a point (p, q) on the line $i = p$. The point $(p + 1, q + 2k + 1)$ on the line $i = p + 1$ will have the same colour as (p, q). □

From Lemma 5.1, we see that a colouring for $c(\sigma)$ points along a line $i = m$ for some arbitrary m describes the colouring for the entire grid.

To colour along the line $i = 0$, we proceed as follows: Starting with the point $(0, 0)$ which is assigned the colour 0, we assign consecutive colours to every second vertex, and wrap around after the $c(\sigma)$th vertex. This will colour all the $c(\sigma)$ points in two passes uniquely. This can be easily seen because $c(\sigma) = c'(k) = 2k^2 + 2k + 1$ is odd and hence, points coloured in the first pass will not be repeated again. Consecutive sets of $c(\sigma)$ vertices along this line follow the same colouring pattern.

Mathematically, this colouring scheme can be expressed as follows. Let $\Omega(i, j)$ represent the colour assigned to the vertex (i, j) and $\Omega'(j)$ represent the colour assigned to the vertex $(0, j)$, i.e. $\Omega'(j) = \Omega(0, j)$. We first give a formula for $\Omega'(j)$ and then derive an expression for $\Omega(i, j)$.

$$\Omega'(j) = \begin{cases} \rho, & \bar{j} \text{ even} \\ \rho + k^2 + k + 1, & \bar{j} \text{ odd} \end{cases}$$

where $\bar{j} = j \bmod c(\sigma)$ and $\rho = \lfloor \frac{\bar{j}}{2} \rfloor$. Now, from Lemma 5.3, we can easily derive that

$$\Omega(i, j) = \Omega'\big(j - i(2k+1)\big)$$

From the above formula, it can be easily seen that given any arbitrary point (i, j) in the grid, the colour assigned to (i, j) can be computed in constant time.

Theorem 4. *For all $\sigma = 2k+1, k = \{1, 2, \ldots\}$, the colouring scheme described above is an optimal $L(\delta_1, \vec{1}_{\sigma-2})$-colouring for $\mathcal{G}(\mathbf{Z}^2)$, with $\delta_1 = k^2$.*

Proof. From Lemma 2, $c(\sigma)$ is a lower bound. We can easily see that each vertex in the tile is assigned a unique colour from the set $\{0, 1, \ldots, c(\sigma) - 1\}$. This implies that the optimality condition is satisfied.

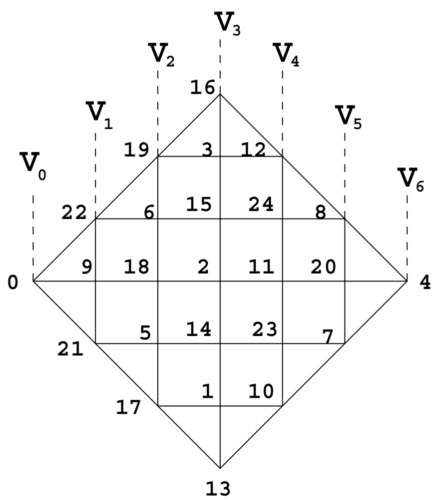

Figure 10. $L(\delta_1, \vec{1}_{\sigma-2})$ **colouring for** $\sigma = 7$

Again, the above scheme ensures that corresponding points in neighboring tiles have the same colour and are exactly σ distance apart. Thus, the re-use constraint is satisfied.

To derive the value of δ_1, we proceed as follows. Consider a point (i, j) in the grid. Its four neighbors are $(i-1, j)$, $(i, j+1)$, $(i+1, j)$, and $(i, j-1)$.

The colour assigned to (i, j) according to the above scheme will be $\Omega(i, j) = \Omega'\big(j - i(2k+1)\big)$.

The following table shows the colours (modulo $c(\sigma)$) assigned to the neighbors of (i, j).

$(i-1, j)$	$\big(\Omega(i, j) + k^2 + 2k + 1\big)$
$(i, j+1)$	$\big(\Omega(i, j) + k^2 + k + 1\big)$
$(i+1, j)$	$\big(\Omega(i, j) + k^2\big)$
$(i, j-1)$	$\big(\Omega(i, j) - k^2 - k - 1\big)$

From the above table, we see that the least difference between the colours assigned to neighbouring points is k^2. Hence, $\delta_1 = k^2$. □

Even σ

We now present a colouring scheme for even $\sigma, \sigma \geq 4$, i.e. $\sigma = 2k$, $k \in \{2, 3, \ldots\}$. We first note that the total number of points in a tile in terms of k will be equal to $2k^2$. Since colouring of the entire grid is achieved by colouring one tile and reproducing the same colouring in all tiles of the grid, description of the colouring for a single tile is sufficient.

The colouring scheme is shown in Figure 11. Alternate diagonals are coloured consecutively starting with D_0, i.e. the following diagonals D_0, D_2, $\ldots$, $D_{\sigma-2}$, D_1, D_3, $\ldots$, $D_{\sigma-1}$ are coloured in order. Starting with the origin of the tile which is assigned colour 0, points are coloured consecutively within each diagonal.

Let $\Omega(i, j)$ be the colour assigned to the point (i, j) in the grid. It can be mathematically expressed as follows:

$$\begin{aligned}\Omega(i, j) \quad = \quad & \lfloor \frac{(i-j) \bmod 2k}{2} \rfloor k + \\ & \lfloor \frac{(i+j) \bmod 2k}{2} \rfloor \\ & + ((i+j) \bmod 2)k^2\end{aligned}$$

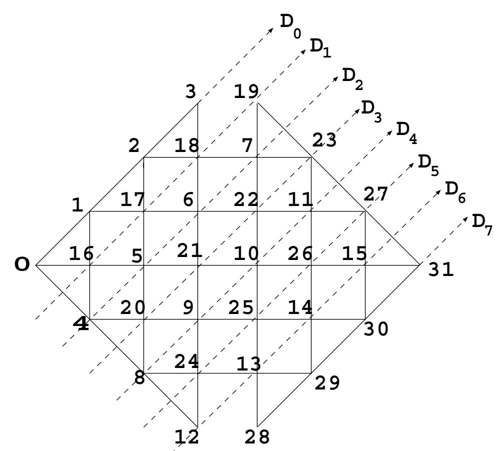

Figure 11. $L(\delta_1, \vec{1}_{\sigma-2})$ **colouring for** $\sigma = 8$

Theorem 5. *For all $\sigma = 2k, k = \{2, 3, \ldots\}$, the colouring scheme described above is an optimal $L(\delta_1, \vec{1}_{\sigma-2})$-colouring for $\mathcal{G}(\mathbf{Z}^2)$, with $\delta_1 = k^2 - k - 1$.*

Proof. From Lemma 2, $c(\sigma)$ is a lower bound. We can easily see that each vertex in the tile is assigned a unique colour from the set $\{0, 1, \ldots, c(\sigma) - 1\}$. This implies that the optimality condition is satisfied.

Again, from the formula, we see that corresponding points in neighboring tiles have the same colour and are exactly σ distance apart. Thus, the re-use constraint is satisfied.

To derive the value of δ_1, we proceed as follows. Consider the neighbors of an arbitrary point (i, j) in the grid. They are $(i-1, j)$, $(i, j+1)$, $(i+1, j)$ and $(i, j-1)$. We will find the differences between the colours assigned to (i, j) and each of its neighbors. The least difference will be equal to δ_1.

There are two cases to consider: 1) $(i+j)$ is even. 2) $(i+j)$ is odd. But note that $(i+j)$ value for alternate points in both X and Y directions will be of the same parity. If we consider a point for which $(i+j)$ is odd, $(i+j)$ for all its neighbors will be even and vice versa. It follows that we need to consider only one case, considering the other case too will yield the same expressions for the differences.

Consider a point (i, j) and suppose $(i+j)$ is odd. Let the colour assigned to (i, j) be $\Omega(i, j)$. The following table shows the colours assigned to the neighbors of (i, j).

$(i-1, j)$	$\Omega(i,j) - k^2$
$(i, j+1)$	$\Omega(i,j) - k^2 - k + 1$, if $(i+j) \bmod 2k = 2k-1$
	$\Omega(i,j) - k^2 + 1$, otherwise
$(i+1, j)$	$\Omega(i,j) - k^2 + 1$, if $(i+j) \bmod 2k = 2k-1$
	$\Omega(i,j) - k^2 + k + 1$, otherwise
$(i, j-1)$	$\Omega(i,j) - 2k^2 + k$, if $(i-j) \bmod 2k = 2k-1$
	$\Omega(i,j) - k^2 + k$, otherwise

Clearly, from the above table, the least difference between the colours assigned to neighbouring points is $k^2 - k - 1$. Hence, $\delta_1 = k^2 - k - 1$. □

3.3 Tight Upper Bound on δ_1

The previous subsections presented colouring schemes for odd reuse distances where δ_1, the channel separation constraint had a value of k^2, where $\sigma = 2k+1$. Based on experimental verification by means of an exhaustive search for all values of $k \leq 4$, we conjecture that k^2 is, in fact, an upper bound on the value of δ_1 for optimal colourings of square and cellular grids with odd reuse distance $\sigma = 2k+1$.

4 Conclusions

We characterised optimal channel assignment schemes for cellular and square grids, and hence showed that any such scheme must be uniform across the entire grid. More specifically, in an optimal colouring, the colouring of a tile (for a given σ) will be identically repeated in all the tiles throughout the grid. We also presented optimal $L(\delta_1, \vec{1}_{\sigma-2})$ colouring schemes, with a high value for δ_1, for square grids for all $\sigma \geq 4$ and for cellular grids for the case where *reuse distance* is odd i.e., $\sigma = 2k+1$, $k \in \{1, 2, \ldots\}$. The previous best known results have been restricted to $\delta_1 \lessapprox \frac{8k^2}{3}$ [3, 4], in case of cellular grids and $\delta_1 \leq \lfloor \frac{\sigma-1}{2} \rfloor$ [2] in case of square grids. We conjecture that our value of δ_1 is a tight upper bound on δ_1 for optimal colouring schemes for these grids.

Several interesting open questions arise from the work presented here. We list a few of them here: (1) Find optimal colouring schemes for cellular grids with high δ_1 values for the case when σ is even. (2) Find and prove the existence of tight upper bounds for $\delta_1, \delta_2, \ldots$ for a general σ.

References

[1] A. A. Bertossi, C. M. Pinotti, and R. B. Tan. Efficient use of radio spectrum in wireless networks with channel separation between close stations. In *Proceedings of the DIAL M Workshop*, pages 18–27, 2000.

[2] A. A. Bertossi, C. M. Pinotti, and R. B. Tan. Channel assignment with separation for special classes of wireless networks: grids and rings. In *Proceedings of IPDPS*, 2002.

[3] A. Dubhashi, A. Pati, M. V. S. Shashanka, S. R., and A. M. Shende. Channel assignment in wireless networks modelled as cellular and d-dimensional square grids. http://citeseer.nj.nec.com/, 2002.

[4] A. Dubhashi, A. Pati, M. V. S. Shashanka, R. Shashank, and A. M. Shende. Channel assignment for wireless networks modelled as d-dimensional square grids. In S. K. Das and S. Bhattacharya, editors, *Proceedings of the 4th International Workshop on Distributed Computing*, number 2571 in LNCS, pages 130–141, Calcutta, India, December 2002. Springer Verlag.

[5] W. Hale. Frequency assignment: Theory and application. *Proceedings of the IEEE*, 68:1497–1514, 1980.

[6] R. A. Murphey, P. M. Pardalos, and M. G. Resende. Frequency assignment problems. In *Handbook of Combinatorial Optimization*. Kluwer Academic Press, 1999.